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14. ABSTRACT
Symbolic equations for the effective continuum stiffness and strength properties of several periodic beam-like trusses have been previously derived and are well documented in the literature. These equations are useful because they allow for rapid design and assessment of structures that would otherwise require a more time-consuming analysis. Previous investigations have considered changes in truss construction, such as the number of longerons and diagonal lacing as discrete cases; unique sets of equations were derived for each unique construction. These equations did not restrict the relative sizes of longerons, diagonals and battens. In the present work, a generic set of equations is derived that is applicable to trusses with an arbitrary numbers of longerons and diagonal lacings, however, the diagonals must be soft relative to the longerons and battens. The resulting equations are useful in preliminary truss sizing and optimization routines because they allow the number of longerons and diagonals to be changed by simply changing the value of a constant in the equations. In this paper, equations are derived for effective continuum beam bending, torsion, shear and axial loading. Within the assumption of relatively soft diagonals, the equations are shown to be equivalent to the three, four and six longeron results previously published by Renton and are numerically verified through comparison to finite element analysis solutions.

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Symbolic Equations for the Stiffness and Strength of Straight Longeron Trusses

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Symbolic equations for the effective continuum stiffness and strength properties of several periodic beam-like trusses have been previously derived and are well documented in the literature. These equations are useful because they allow for rapid design and assessment of structures that would otherwise require a more time-consuming analysis. Previous investigations have considered changes in truss construction, such as the number of longerons and diagonal lacing as discrete cases; unique sets of equations were derived for each unique construction. These equations did not restrict the relative sizes of longerons, diagonals and battens. In the present work, a generic set of equations is derived that is applicable to trusses with an arbitrary numbers of longerons and diagonal lacings, however, the diagonals must be soft relative to the longerons and battens. The resulting equations are useful in preliminary truss sizing and optimization routines because they allow the number of longerons and diagonals to be changed by simply changing the value of a constant in the equations. In this paper, equations are derived for effective continuum beam bending, torsion, shear and axial loading. Within the assumption of relatively soft diagonals, the equations are shown to be equivalent to the three, four and six longeron results previously published by Renton and are numerically verified through comparison to finite element analysis solutions.

Nomenclature

Symbols

A	=	element cross-section area (m^2),
c	=	truss shear coefficient,
E	=	Young's modulus (N/m^2),
G	=	shear modulus (N/m^2),
I	=	cross-section moment of inertia (m^4),
J	=	cross-section polar moment of inertia (m^4),
L	=	truss length (m),
l	=	element length (m),
M	=	bending moment (Nm),
n	=	number of longerons,
P	=	axial load (N),
R	=	truss radius (m),
T	=	torsion load (Nm),
V	=	truss or face shear load (N),
x	=	longeron distance from x axis (m),
δ	=	lateral deformation (m),
κ	=	curvature (1/m),
θ	=	truss diagonal angle (rad),
ϕ	=	truss twist about its long axis (rad),

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ψ	=	longeron angular position (rad),
ε	=	extensional strain,
γ	=	shear strain,
ϑ	=	truss twist per length (rad/m),
η	=	number of diagonals per truss face, 1 or 2

Subscripts

b	=	batten
c	=	face center
d	=	diagonal
f	=	face
i	=	longeron index number
l	=	longeron

I. Introduction

Preliminary design and analysis of beam-like trusses is simplified by concise symbolic equations for their effective continuum behavior. Fortunately, these equations are readily attainable for contemporary truss designs. Cost and the need for design simplicity typically drive trusses to be periodic; they are built up from identical repeating cell constructions *ad infinitum*. (A *construction* here defines a specific arrangement of structural elements within a repeating cell, for example, the number of longerons. *Sizing* refers to the cross-section and length of each element.) The elastic behavior of periodic trusses can be predicted based on the elastic behavior of the repeating cell. Repeating cells have simple constructions (a four longeron truss with two diagonals per face has 16 elements per cell, only three of which are unique), making them conducive to closed form symbolic equation description. Beam-like trusses built up from a linear (end-to-end) stacking of these cells are similarly analytically simple when analyzed as an effective continuum.

Symbolic equations for several periodic truss constructions have been previously derived and will be reviewed in the first section of this paper. In these works, the equations were derived for a specific repeating cell construction; any change in this construction requires derivation of a new set of effective continuum equations. This paper expands on the previous work by deriving stiffness and strength equations for trusses with an arbitrary number of straight longerons tied together with perpendicular batten frames and an arbitrary diagonal lacing pattern. While these equations do not assess global elastic stability (buckling), they allow the truss elements to be designed to rationally derived tensile and compressive loads. The simplified stiffness and strength equations presented here accurately model trusses with three or more longerons (such as coilable longeron masts and articulated masts) as well as isogrids tubes with slender elements. The equations assume the elements are two force pinned elements (they do not consider element bending) and are restricted to trusses where the diagonals are much softer than the longerons.

II. Previous Effective Continuum Equation Derivations

Renton[1-3] and Noor[4] derived expressions for the effective continuum stiffness behavior of several trusses. In terms of the truss radius (the radius of the circle on which the longerons lie, R) and diagonal angle (angle from the batten frame to the diagonal, θ), these equations for a three longeron truss with two diagonals per face are,

$$\begin{aligned}
EA &= 3EA_l + \frac{6EA_bEA_d\sin^3\theta}{EA_b + 2EA_d\cos^3\theta} \\
EI &= \frac{3}{4}R^2\left(2EA_l + \frac{EA_bEA_d\sin^3\theta}{EA_b + 2EA_d\cos^3\theta}\right) \\
GA &= \frac{3EA_d\tan\theta\left(2EA_bEA_l\cot^3\theta + EA_d\cos^3\theta\left(EA_b + 4EA_l\cot^3\theta\right)\right)^2}{6EA_bEA_d^3\cos^6\theta + EA_d^2\cos^3\theta\left(EA_b + 4EA_l\cot^3\theta\right)^2 + 4EA_bEA_l\cot^3\theta\left(4EA_dEA_l\cot^3\theta + EA_b\left(EA_d + EA_l\csc^3\theta\right)\right)} \\
GJ &= \frac{3}{2}EA_dR^2\cos^2\theta\sin\theta
\end{aligned} \quad (1)$$

The equations for a three longeron truss with only one diagonal per face and where these diagonals are laced such that they all spiral the same direction within a bay and alternated directions between bays are,

$$
\begin{aligned}
EA &= 3EA_l \\
EI &= \frac{3}{2}EA_l R^2 \\
GA &= \frac{6EA_l EA_d \tan\theta}{4EA_l \sec^3\theta + EA_d \tan^3\theta} \\
GJ &= \frac{3EA_l EA_d R^2 \tan\theta}{4\left(EA_l \sec^3\theta + EA_d \tan^3\theta\right)}
\end{aligned} \tag{2}
$$

The corresponding equations for a four longeron truss with two diagonals per face are,

$$
\begin{aligned}
EA &= 4\left(EA_l + 2EA_d \sin^3\theta\right) \\
EI &= 2R^2\left(EA_l + \frac{EA_b EA_d \sin^3\theta}{EA_b + 2EA_d \cos^3\theta}\right) \\
GA &= \frac{4EA_d\left(EA_b EA_l \cot^3\theta + EA_d \cos^3\theta\left(EA_b + 2EA_l \cot^3\theta\right)\right)^2 \tan\theta}{2EA_b EA_d^3 \cos^6\theta + EA_d^2 \cos^3\theta\left(EA_b + 2EA_l \cot^3\theta\right)^2 + EA_b EA_l \cot^3\theta\left(2EA_b EA_d + 4EA_d EA_l \cot^3\theta + EA_b EA_l \csc^3\theta\right)} \\
GJ &= 4EA_d R^2 \cos^2\theta \sin\theta
\end{aligned} \tag{3}
$$

Infinitely stiff cross brace diagonals, located in the plane of each batten frame, have been assumed to stabilize the square batten frames from shear deformations. Finally, the equations for a four longeron truss with one diagonal per face and laced such that the diagonals spiral in the same direction in every face are,

$$
\begin{aligned}
EA &= 4EA_l \\
EI &= 2EA_l R^2 \\
GA &= \frac{4EA_l EA_d EA_b \tan\theta}{2EA_l\left(EA_d + EA_b \sec^3\theta\right) + EA_d EA_b \tan^3\theta} \\
GJ &= \frac{2EA_l EA_d EA_b R^2 \tan\theta}{EA_l EA_b \sec^3\theta + EA_d\left(EA_l + EA_b \tan^3\theta\right)}
\end{aligned} \tag{4}
$$

These equations are simplified by considering the relative axial stiffness of longerons, diagonals and battens and how they are loaded. Trusses are typically sized such that the longeron axial stiffness is much greater than the diagonal and sometimes the batten stiffness. With truss axial and bending loading, the longerons become highly loaded while the battens and diagonals, due to geometry and their lower stiffness, are only minimally stressed. As a result, longeron compliance typically dominates EA and EI. In torsion and shear loading configurations, the diagonals become highly stressed while longerons and battens are somewhat less stressed. The diagonals are typically much more compliant than the longerons so that GA and GJ are dominated by the diagonal compliance. Formally, these assumptions are 1) that either the diagonal or batten stiffness approach zero for EA and EI and 2) that both the longeron and the batten stiffness approach infinity for GA and GJ,

$$
\begin{aligned}
&\left.\begin{matrix} EA \\ EI \end{matrix}\right\} EA_d \to 0 \text{ or } EA_b \to 0 \\
&\left.\begin{matrix} GA \\ GJ \end{matrix}\right\} EA_l \to \infty \text{ and } EA_b \to \infty
\end{aligned} \tag{5}
$$

Implementing these limits in Equation (1), a three longeron truss with two diagonals per face, yields,

$$
\begin{aligned}
EA &= 3EA_l \\
EI &= \frac{3}{2}EA_l R^2 \\
GA &= 3EA_d \cos^2\theta \sin\theta \\
GJ &= \frac{3}{2}EA_d R^2 \cos^2\theta \sin\theta
\end{aligned} \tag{6}
$$

A similar procedure can be followed for the other truss constructions. The simplified equations for three longeron trusses with one diagonal per face are identical to Equations (6) except that GA and GJ are multiplied by ½,

$$
\begin{aligned}
EA &= 3EA_l \\
EI &= \frac{3}{2} EA_l R^2 \\
GA &= \frac{3}{2} EA_d \cos^2\theta \sin\theta \\
GJ &= \frac{3}{4} EA_d R^2 \cos^2\theta \sin\theta
\end{aligned} \tag{7}
$$

The four longeron truss construction with two diagonals per face yields the following simplified equations,

$$
\begin{aligned}
EA &= 4EA_l \\
EI &= 2EA_l R^2 \\
GA &= 4EA_d \cos^2\theta \sin\theta \\
GJ &= 4EA_d R^2 \cos^2\theta \sin\theta
\end{aligned} \tag{8}
$$

Four longeron trusses with only one diagonal per face are similarly identical to Equations (8) except that GA and GJ should be multiplied by ½,

$$
\begin{aligned}
EA &= 4EA_l \\
EI &= 2EA_l R^2 \\
GA &= 2EA_d \cos^2\theta \sin\theta \\
GJ &= 2EA_d R^2 \cos^2\theta \sin\theta
\end{aligned} \tag{9}
$$

Finally, Renton also derived the equations for six longeron trusses. After simplification, the equations for two diagonals per face are,

$$
\begin{aligned}
EA &= 6EA_l \\
EI &= 3EA_l R^2 \\
GJ &= 9EA_d R^2 \cos^2\theta \sin\theta
\end{aligned} \tag{10}
$$

Renton did not derive the equations for GA for six longeron trusses in Reference 1. The simplified equations for six longeron trusses with one diagonal per face are identical to Equations (10) except that GJ is multiplied by ½,

$$
\begin{aligned}
EA &= 6EA_l \\
EI &= 3EA_l R^2 \\
GJ &= \frac{9}{2} EA_d R^2 \cos^2\theta \sin\theta
\end{aligned} \tag{11}
$$

III. Current Derivations: Stiffness

In the current derivations, simplified truss equations similar to Equations (6) through (11), are sought as a function of the number of longerons (n) and the number of diagonals per face (η , either 1 or 2). Consider the truss construction to be of n evenly spaced straight longerons, all of which lie on a circle of radius R . Perpendicular to the longerons are planar batten frames where each batten is of length,

$$
l_b = 2R \sin\left(\frac{\pi}{n}\right) \tag{12}
$$

The batten frames are evenly distributed along the longerons at an interval (bay length) of,

$$
l_l = 2R \sin\left(\frac{\pi}{n}\right) \tan\theta \tag{13}
$$

The resulting diagonal lengths are,

$$
l_d = \frac{2R}{\cos\theta} \sin\left(\frac{\pi}{n}\right) \tag{14}
$$

The relations between longeron, diagonal and batten length are also useful,

$$l_l = l_b \tan\theta$$
$$l_l = l_d \sin\theta \qquad (15)$$
$$l_b = l_d \cos\theta$$

Truss axial stiffness is the simplest case and is considered first. Because primarily the longerons are loaded, it is reasonably intuitive to assume that the axial stiffness of a truss is the sum of the axial stiffnesses of all the longerons,

$$EA = nEA_l \qquad (16)$$

Derivation of the truss bending stiffness results in a similarly simple expression. Assume the first longeron is located at an angle of ψ_1 from the x axis. Subsequent longerons are located at,

$$\psi_i = \psi_1 + \frac{2\pi}{n}(i-1) \qquad (17)$$

ψ_1 results in unique solutions from $\psi_1 = 0$ to $\psi_1 = \pi/n$; values outside this range result in redundant longeron positions. Assume the truss bends about the truss geometric center so that the distance from the y axis to a specific longeron is,

$$x_i = R\cos\psi_i \qquad (18)$$

The bending stiffness of the truss is the sum of the contribution from each longeron,

$$EI = \sum_{i=1}^{n} EA_l x_i^2 = EA_l R^2 \sum_{i=1}^{n} \cos^2\left(\psi_1 + \frac{2\pi}{n}(i-1)\right) = \frac{n}{2} EA_l R^2 \qquad (19)$$

ψ_1 is not explicit in the final expression showing that EI is isotropic; truss bending stiffness is the same in every direction. Equation (19) was previously derived in Reference 5.

The torsional and shear stiffness equations are derived by considering the shear stiffness of one truss face, as shown in Figure 1.

$$\frac{V_f}{\delta_f} = \eta \frac{EA_d}{2R} \csc\left(\frac{\pi}{n}\right)\cos^3\theta \qquad (20)$$

The shear stiffness of a truss is related to this face stiffness by,

$$GA = c\frac{V_f}{\delta_f / l_l} \qquad (21)$$

where c is a coefficient to correct for the number faces and the orientation of each face within a bay. The coefficient includes two cosine reduction factors 1) because each face is not oriented parallel to the shear direction and 2) because each face deformation direction is not parallel to the face. The resulting coefficient is,

$$c = \sum_{i=1}^{n} \cos^2\psi_i = \frac{n}{2} \qquad (22)$$

The truss shear stiffness is then,

$$GA = \eta\frac{n}{2} EA_d \cos^2\theta \sin\theta \qquad (23)$$

As with bending stiffness, the shear stiffness is isotropic.

The truss torsional stiffness is n times the torsional stiffness provided by a single face,

$$GJ = n\frac{T_f}{\phi / l_l} \qquad (24)$$

where T_f is the moment contribution from a single face in a truss twisted by an angle of ϕ. The line of action of the moment occurs at the center of the face, which is closer to the truss rotational center by $\cos\left(\pi/n\right)$. T_f and ϕ are thus expressed as,

$$T_f = V_f R\cos\left(\frac{\pi}{n}\right) \qquad (25)$$

$$\phi = \frac{\delta_f}{R\cos\left(\frac{\pi}{n}\right)} \qquad (26)$$

Substitution of Equations (20), (25) and (26) into Equation (24) gives the truss torsional stiffness,

$$GJ = \eta n E A_d R^2 \cos^2\left(\frac{\pi}{n}\right)\cos^2\theta\sin\theta \tag{27}$$

The current truss derivations are summarized in Table 1. By inspection, it is apparent they are identical to the simplified forms of Renton's derivations from Equations (6) through (11).

IV. Current Derivations: Element Loads

Element loads are calculated in a systematic process that takes advantage of the truss stiffness results of the previous section. First, the truss stiffness equations are used to calculate the element strain that results from a truss load. Second, element loads that result from this strain are calculated. Within the assumptions of Equation (5), it is not necessary to account for elasticity in calculating element loads; however, the employed process allows the assumptions of soft diagonals and stiff longerons to be more rigorously tracked than otherwise convenient.

Truss axial, bending, shear and torsional stiffness are defined as,

$$EA = \frac{P}{\varepsilon},\ \ EI = \frac{M}{\kappa},\ \ GA = \frac{V}{\gamma},\ \ GJ = \frac{T}{\vartheta} \tag{28}$$

P is a truss axial load and ε is the resulting axial change in length per length of truss. M is a truss bending moment and κ is the change in truss angle per length of truss and is equivalent to the truss curvature. V is a uniform truss shear load and γ is the resulting shear deflection per truss length. T is a truss torsion load and ϑ is the truss twist per length. It is difficult to directly measure the shear stiffness of a truss because shear loads are coupled to moments and the resulting deflections are dependent on EI as well as GA. Fortunately, EI is readily isolated through application of pure moments so that for a cantilever beam, GA is,[6]

$$GA = VL\left(\delta - \frac{VL^3}{3EI}\right)^{-1} \tag{29}$$

Assuming the truss elements are pinned at both ends so that they only carry axial loads, the loads in a longeron, diagonal and batten as a function of their strain are,

$$P_l = \varepsilon_l E A_l,\ \ P_d = \varepsilon_d E A_d,\ \ P_b = \varepsilon_b E A_b \tag{30}$$

The strength of a truss with axial loading is the simplest case and is considered first. Assuming the longerons are much stiffer than the diagonals, a truss axial load only significantly loads the longerons. When a diagonal load does result, it is typically much smaller than that derived from a shear or torsion load. The longeron strain that results from a truss axial strain is identical to the truss axial strain, $\varepsilon_l = \varepsilon$. Combining this and Equations (16), (28) and (30), the ratio of element to truss load is,

$$\frac{P_l}{P} = \frac{1}{n} \tag{31}$$

A truss loaded in bending similarly only loads the longerons when the longerons are much stiffer than the diagonals. Combining Equations (19) and (28), the truss bending moment to curvature ratio is,

$$\frac{M}{\kappa} = \frac{n}{2} E A_l R^2 \tag{32}$$

Assuming the truss bends about its geometric center, truss curvature as a function of longeron strain is,

$$\kappa = \frac{\varepsilon_l}{R\cos\theta_1} \tag{33}$$

where θ_1 is the position of the first longeron. Combining Equations (30), (32) and (33), the longeron load to bending moment ratio is,

$$\frac{P_l}{M} = \frac{2\cos\theta_1}{nR} \tag{34}$$

Truss bending strength is minimum when the longeron with the largest compressive load ($i = 1$) is farthest from the neutral axis; i.e. when $\theta_1 = 0$,

$$\frac{P_{l,\max}}{M} = \frac{2}{nR} \tag{35}$$

The truss bending strength is maximum when the longeron with the largest compressive load is closest to the neutral axis; i.e. when $\theta_1 = \pi/n$,

$$\frac{P_{l,\min}}{M} = \frac{2}{nR}\cos\left(\frac{\pi}{n}\right) \tag{36}$$

The maximum and minimum bending strengths differ by a factor of $\cos\left(\pi/n\right)$, which is $1/2$ for $n = 3$ and approaches 1 as the number of longerons increases.

A truss loaded in torsion results in a constant and uniform shear deformation in each truss face. Combining Equations (27) and (28), the truss twist per length that results from a torsional load is,

$$\frac{T}{\vartheta} = \eta n E A_d R^2 \cos^2\left(\frac{\pi}{n}\right)\cos^2\theta\sin\theta \tag{37}$$

Diagonal length change as a function of truss face shear displacement is given by,

$$\delta_d = \delta_f \cos\theta \tag{38}$$

Combining Equation (26) with Equations (15) and (38), the truss twist per length as a function of diagonal strain is,

$$\vartheta = \frac{\phi}{l_l} = \frac{\varepsilon_d}{R\cos\theta\sin\theta\cos\dfrac{\pi}{n}} \tag{39}$$

where $\varepsilon_d = \delta_d/l_d$. Combining Equations(30), (37), and (39) yields the ratio of diagonal load to truss load,

$$\frac{P_d}{T} = \frac{1}{\eta n R\cos\theta\cos\dfrac{\pi}{n}} \tag{40}$$

A truss loaded in shear results in a face shear and diagonal load that depends on the orientation of the face relative to the shear load. Combing Equations (23) and (28), the truss shear deflection per length that results from a shear load is,

$$\frac{V}{\gamma} = \eta\frac{n}{2}EA_d\cos^2\theta\sin\theta \tag{41}$$

Diagonal length change as a function of truss shear depends on the orientation of the truss face. Let the angle between the direction of the shear load and the plane of a shear face be given by,

$$\psi_c = \frac{\psi_1 + \psi_2}{2} = \psi_1 + \frac{\pi}{n} \tag{42}$$

With this convention, the maximum face shear deformation and hence, diagonal load, occurs when $\psi_1 = -\pi/n$ so that $\psi_c = 0$. The face shear relative to the shear direction is,

$$\delta_f = \frac{\delta}{\cos\psi_c} \tag{43}$$

Combining Equations (15), (38) and (43), the truss shear deformation per length is,

$$\gamma = \frac{\delta}{l_l} = \frac{\varepsilon_d}{\cos\psi_c\cos\theta\sin\theta} \tag{44}$$

Combining Equations (30), (41) and (44) yields the ratio of diagonal load to truss shear load,

$$\frac{P_d}{V} = \frac{2\cos\psi_c}{n\eta\cos\theta} \tag{45}$$

The truss shear strength is minimum when the truss face is oriented so that is receives the maximum shear strain, when $\psi_c = 0$,

$$\frac{P_{d,\max}}{V} = \frac{2}{n\eta\cos\theta} \tag{46}$$

The truss shear strength is maximum when the truss face is oriented so that it receives the minimum shear strain, when $\psi_c = \pi/n$,

$$\frac{P_{d,\min}}{V} = \frac{2\cos\dfrac{\pi}{n}}{n\eta\cos\theta} \tag{47}$$

In trusses with two diagonals per face, loads are cancelled such that shear and torsion loads do not induce loads in the batten and diagonals. This cancelling effect does not occur in trusses with a single diagonal per face and shear and torsion loads induce diagonal and batten loads. These loads are a result of static equilibrium of the joints and are thus simple to calculate. A diagonal load will cause longeron and batten loads of,

$$\begin{aligned} \frac{P_l}{P_d} &= \sin\theta \\ \frac{P_b}{P_d} &= \cos\theta \end{aligned} \tag{48}$$

Using these equations, the longeron and batten loads that result from a torsion load are,

$$\begin{aligned} \frac{P_l}{T} &= \frac{P_d}{T}\frac{P_l}{P_d} = \frac{\tan\theta}{nR\cos\dfrac{\pi}{n}} \\ \frac{P_b}{T} &= \frac{P_d}{T}\frac{P_b}{P_d} = \frac{1}{nR\cos\dfrac{\pi}{n}} \end{aligned} \tag{49}$$

Similarly, the loads that result from a shear load are,

$$\begin{aligned} \frac{P_{l,\max}}{V} &= \frac{P_{d,\max}}{V}\frac{P_l}{P_d} = \frac{2}{n}\tan\theta \\ \frac{P_{b,\max}}{V} &= \frac{P_{d,\max}}{V}\frac{P_b}{P_d} = \frac{2}{n} \\ \frac{P_{l,\min}}{V} &= \frac{P_{d,\min}}{V}\frac{P_l}{P_d} = \frac{2}{n}\cos\frac{\pi}{n}\tan\theta \\ \frac{P_{b,\min}}{V} &= \frac{P_{d,\min}}{V}\frac{P_b}{P_d} = \frac{2}{n}\cos\frac{\pi}{n} \end{aligned} \tag{50}$$

V. Comparison with Finite Element Solutions

The truss stiffness and strength equations derived in this paper were shown to be accurate through comparison with finite element analysis solutions generated with Abaqus (ABAQUS, Inc.), a general purpose finite element analysis program. Abaqus Python scripts were written to build and solve models of trusses with an arbitrary number of longerons, bays and diagonal lacing. These solutions were then interrogated for effective stiffnesses and element loads. Representative models are shown in Figure 2. Three reference trusses were considered and are listed in Table 2. Four relative arrangements of elements axial stiffnesses were also considered and are listed in Table 3.

Finite element analysis and simplified symbolic equation stiffness results are compared in Table 4 for one diagonal per face and in Table 5 for two diagonals per face. For one diagonal per face the equations are exact for EA and EI, regardless of the truss configuration. The equations for GA and GJ are accurate within 0.76% for truss configuration C4 (diagonals are 100 times more compliant than longerons and battens) and are accurate within 7.57% for truss configuration C2 (diagonals are 10 times more compliant than longerons and battens). The equations are in error by as much as 75.7% for truss configuration C1, where all elements have the same cross section. For trusses with two diagonals per face, only the equation for GJ is exact for all truss configurations. For truss configuration C4, the equations are accurate to within 1.16%. For truss configuration C2, the equations are accurate to within 10.3%. For truss configuration C1, the equations are in error by as much as 55.4%.

Finite element analysis and simplified symbolic equation strength results are compared in Table 6 for one diagonal per face in all truss configurations, in Table 7 for configuration C1 with two diagonals per face and in Table 8 for configuration C4 with two diagonals per face. The equations are precise for one diagonal per face in all configurations. This is because trusses with one diagonal per face are statically determinant and hence, the element loads are not a function of the truss elasticity. Some error occurs in the equations for two diagonals per face. In truss configuration C4, the load equations are accurate to within 1.18%. In configuration C1, the equations are in error by as much as 89.2%.

The equations in Table 1 were also published in Chapter 1 of Reference 7, however, the details of their source and derivation were not discussed.

VI. Conclusion

The conclusions of this work are rather mundane since no additional insight into the behavior of trusses has been revealed. The utility in the work is in having a concise reference to find simple equations to quickly evaluate the stiffness and element loads for a large class of trusses.

However, one can glean insights from the results. For example, the equations rigorously elucidate the scaling of truss stiffness and strength with number of longeron. Truss axial, bending and shear stiffness scale linearly with the number of longerons. Truss torsional stiffness scales with,

$$n\cos^2\left(\frac{\pi}{n}\right) \tag{51}$$

Presumably the cosine term occurs because as the number of longerons is increased, the truss more closely approximates a tube by placing the diagonals at the greatest effective radius. Maximum element loads simply scale with the inverse of the number of longerons. Minimum element loads scale similarly, with the additions of a cosine term,

$$\cos\left(\frac{\pi}{n}\right) \tag{52}$$

that accounts for the placement of elements relative to the axis of bending. As would be expected, at higher numbers of longerons there is little difference between the maximum and minimum element loads.

The equations also illustrate the interesting fact that shear stiffness and strength are not a function of truss radius. While increasing the truss radius will dramatically increase bending and torsional strength and stiffness, it does not change the truss shear properties. This implies that a stout truss loaded in shear does not benefit from its increased structural depth.

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Table 1: Truss stiffness and strength equations assuming longerons are much stiffer than diagonals.

	Stiffness	Load Ratio with 1 Diagonal Per Truss Face	Load Ratio with 2 Diagonals Per Truss Face
Axial	$EA = nEA_l$	$\frac{P_l}{P} = \frac{1}{n}$ $\frac{P_d}{P} = 0$ $\frac{P_b}{P} = 0$	Same as 1 diagonal per truss face
Bending	$EI = \frac{n}{2} EA_l R^2$	$\frac{P_{l,\max}}{M} = \frac{2}{nR}$, $\frac{P_{l,\min}}{M} = \frac{2}{nR}\cos\left(\frac{\pi}{n}\right)$ $\frac{P_d}{P} = 0$ $\frac{P_b}{P} = 0$	Same as 1 diagonal per truss face
Shear	$GA = \eta \frac{n}{2} EA_d \cos^2\theta \sin\theta$	$\frac{P_{l,\max}}{V} = \frac{2}{n}\tan\theta$, $\frac{P_{l,\min}}{V} = \frac{2}{n}\cos\frac{\pi}{n}\tan\theta$ $\frac{P_{d,\max}}{V} = \frac{2}{n\cos\theta}$, $\frac{P_{d,\min}}{V} = \frac{2\cos\frac{\pi}{n}}{n\cos\theta}$ $\frac{P_{b,\max}}{V} = \frac{2}{n}$, $\frac{P_{b,\min}}{V} = \frac{2}{n}\cos\frac{\pi}{n}$	$\frac{P_{d,\max}}{V} = \frac{1}{n\cos\theta}$ $\frac{P_b}{V} = 0$
Torsion	$GJ = \eta nEA_d R^2 \cos^2\left(\frac{\pi}{n}\right)\cos^2\theta\sin\theta$	$\frac{P_l}{T} = \frac{\tan\theta}{nR\cos\frac{\pi}{n}}$ $\frac{P_d}{T} = \frac{1}{nR\cos\frac{\pi}{n}\cos\theta}$ $\frac{P_b}{T} = \frac{1}{nR\cos\frac{\pi}{n}}$	$\frac{P_l}{T} = 0$ $\frac{P_d}{T} = \frac{1}{2nR\cos\frac{\pi}{n}\cos\theta}$ $\frac{P_b}{T} = 0$

Table 2: Reference truss configurations.

	3 Longeron Truss	4 Longeron Truss	7 Longeron Truss
Number of Longerons, n	3	4	7
Radius, R (m)	0.525	0.471	0.551
Diagonal Angle, θ (deg)	28.0	57.0	34.0

Table 3: Reference truss element properties.

	C1: All Equal	C2: Soft Diagonals	C3: Soft Diagonals and Battens	C4: Very Soft Diagonals
Longeron Axial Stiffness, EA_l (x 10^6 N)	15.0	15.0	15.0	15.0
Diagonal Axial Stiffness, EA_d (x 10^6 N)	15.0	1.50	1.50	0.15
Batten Axial Stiffness, EA_b (x 10^6 N)	15.0	15.0	1.50	15.0

Table 4: Finite element analysis and simplified symbolic equation stiffness comparisons for one diagonal per truss face.

		EA (x 10^6 N)			EI (x 10^6 N-m^2)			GA (x 10^6 N)			GJ (x 10^6 N-m^2)		
		FEA	**Sym**	**% Dif**	**FEA**	**Sym**	**% Dif**	**FEA**	**Sym**	**% Dif**	**FEA**	**Sym**	**% Dif**
$n = 3$	C1	45.00	45.00	0	6.202	6.202	0	4.979	8.235	65.4	0.6458	1.135	75.7
	C2	45.00	45.00	0	6.202	6.202	0	0.7730	0.8235	6.54	0.1055	0.1135	7.57
	C3	45.00	45.00	0	6.202	6.202	0	0.4979	0.8235	65.4	0.6819	1.135	66.4
	C4	45.00	45.00	0	6.202	6.202	0	0.0818	0.0823	0.65	0.0113	0.0113	0.76
$n = 4$	C1	60.00	60.00	0	6.655	6.655	0	5.528	7.463	35.0	0.9497	1.656	74.3
	C2	60.00	60.00	0	6.655	6.655	0	0.7211	0.7463	3.50	0.1541	0.1656	7.43
	C3	60.00	60.00	0	6.655	6.655	0	0.6362	0.7463	17.3	0.1366	0.1656	21.2
	C4	60.00	60.00	0	6.655	6.655	0	0.07437	0.07463	0.35	0.01643	0.01656	0.74
$n = 7$	C1	105.0	105.0	0	15.94	15.94	0	12.07	20.18	67.2	5.795	9.945	71.6
	C2	105.0	105.0	0	15.94	15.94	0	1.891	2.018	6.72	0.9281	0.9945	7.16
	C3	105.0	105.0	0	15.94	15.94	0	1.298	2.0178	55.4	0.6380	0.9945	55.9
	C4	105.0	105.0	0	15.94	15.94	0	0.2004	0.2018	0.67	0.09875	0.09945	0.72

Table 5: Finite element analysis and simplified symbolic equation stiffness comparisons for two diagonals per truss face.

		EA (x 10^6 N)			EI (x 10^6 N-m^2)			GA (x 10^6 N)			GJ (x 10^6 N-m^2)		
		FEA	**Sym**	**% Dif**	**FEA**	**Sym**	**% Dif**	**FEA**	**Sym**	**% Dif**	**FEA**	**Sym**	**% Dif**
$n = 3$	C1	49.08	45.00	-8.32	6.342	6.202	-2.22	18.04	16.47	-8.69	2.270	2.270	0
	C2	45.82	45.00	-1.80	6.230	6.202	-0.46	1.656	1.647	-0.53	0.2270	0.2270	0
	C4	45.09	45.00	-0.20	6.205	6.202	-0.05	0.1648	0.1647	-0.05	0.02270	0.02270	0
$n = 4$	C1	114.2	60.00	-47.5	9.662	6.655	-31.1	33.49	14.93	-55.4	3.311	3.311	0
	C2	66.87	60.00	-10.3	7.036	6.655	-5.4	1.523	1.493	-1.98	0.3311	0.3311	0
	C4	60.71	60.00	-1.16	6.694	6.655	-0.58	0.1496	0.1493	-0.20	0.0331	0.0331	0
$n = 7$	C1	122.8	105.0	-14.5	18.13	15.94	-12.1	43.97	40.36	-8.22	19.89	19.89	0
	C2	108.3	105.0	-3.1	16.35	15.94	-2.50	4.045	4.036	-0.24	1.989	1.989	0
	C4	105.4	105.0	-0.34	15.98	15.94	-0.28	0.4036	0.4036	-0.02	0.1989	0.1989	0

Table 6: Finite element analysis and simplified symbolic equation strength comparisons for one diagonal per truss face (element loads are identical for C1, C2, C3 and C4 truss configurations).

		3 Longeron Truss			4 Longeron Truss			7 Longeron Truss		
		FEA	**Sym**	**% Dif**	**FEA**	**Sym**	**% Dif**	**FEA**	**Sym**	**% Dif**
Axial Load, $P = 1000$ N	P_l	333.3	333.3	0	250.0	250.0	0	142.9	142.9	0
	P_d	0	0	0	0	0	0	0	0	0
	P_b	0	0	0	0	0	0	0	0	0
Bending Moment, $M = 1000$ N-m	$P_{l,\min}$ $P_{l,\max}$	634.9 1270.	634.9 1270.	0 0	750.6 1062.	750.6 1062.	0 0	467.2 518.5	467.2 518.5	0 0
	P_d	0	0	0	0	0	0	0	0	0
	P_b	0	0	0	0	0	0	0	0	0
Shear Load, $V = 1000$ N	$P_{l,\min}$ $P_{l,\max}$	177.2 354.5	177.2 354.5	0 0	544.4 769.9	544.4 769.9	0 0	173.6 192.7	173.6 192.7	0 0
	$P_{d,\min}$ $P_{d,\max}$	377.5 755.0	377.5 755.0	0 0	649.2 918.0	649.2 918.0	0 0	310.5 344.6	310.5 344.6	0 0
	$P_{b,\min}$ $P_{b,\max}$	333.3 666.7	333.3 666.7	0 0	353.6 500.0	353.6 500.0	0 0	257.4 285.7	257.4 285.7	0 0
Torsional Load, $T = 1000$ N-m	P_l	675.2	675.2	0	1156.	1156.	0	194.1	194.1	0
	P_d	1438.	1438.	0	1378.	1378.	0	347.1	347.1	0
	P_b	1270.	1270.	0	750.6	750.6	0	287.8	287.8	0

Table 7: Finite element analysis and simplified symbolic equation strength comparisons for two diagonals per truss face, C1.

		3 Longeron Truss			4 Longeron Truss			7 Longeron Truss		
		FEA	**Sym**	**% Dif**	**FEA**	**Sym**	**% Dif**	**FEA**	**Sym**	**% Dif**
Axial Load, $P = 1000$ N	P_l	306.6	333.3	8.7	132.2	250.0	89.2	122.8	142.9	16.3
	P_d	28.4	0	-	70.26	0	-	17.95	0	-
	P_b	50.2	0	-	76.53	0	-	29.75	0	-
Bending Moment, $M = 1000$ N-m	$P_{l,\min}$ $P_{l,\max}$	621.4 1243	634.9 1270	2.18 2.18	519.2 735.7	750.6 1062	44.6 44.3	412.5 457.8	467.2 518.5	13.3 13.3
	P_d	28.8	0	-	195.2	0	-	54.31	0	-
	P_b	50.9	0	-	212.6	0	-	90.05	0	-
Shear Load, $V = 1000$ N	$P_{d,\max}$	377.5	377.5	0	459.0	459.0	0	172.3	172.3	0
	$P_{b,\max}$	0	0	0	0	0	0	0	0	0
Torsional Load, $T = 1000$ N-m	P_l	0	0	0	0	0	0	0	0	0
	P_d	719.1	719.1	0	689.1	689.1	0	173.6	173.6	0
	P_b	0	0	0	0	0	0	0	0	0

Table 8: Finite element analysis and simplified symbolic equation strength comparisons for two diagonals per truss face, C4.

		3 Longeron Truss			4 Longeron Truss			7 Longeron Truss		
		FEA	**Sym**	**% Dif**	**FEA**	**Sym**	**% Dif**	**FEA**	**Sym**	**% Dif**
Axial Load, $P = 1000$ N	P_l	332.7	333.3	0.20	247.1	250.0	1.18	142.4	142.9	0.35
	P_d	0.7232	0	-	1.732	0	-	0.4402	0	-
	P_b	1.277	0	-	1.887	0	-	0.7298	0	-
Bending Moment, $M = 1000$ N-m	$P_{l,\min}$	634.6	634.9	0.05	746.3	750.6	0.59	465.88	467.2	0.28
	$P_{l,\max}$	1269	1270	0.05	1055	1062	0.59	517.1	518.5	0.28
	P_d	0.6898	0	-	3.700	0	-	1.298	0	-
	P_b	1.218	0	-	4.030	0	-	2.152	0	-
Shear Load, $V = 1000$ N	$P_{d,\max}$	377.5	377.5	0	459.0	459.0	0	172.3	172.3	0
	$P_{b,\max}$	1.020	0	-	5.846	0	-	0.6022	0	-
Torsional Load, $T = 1000$ N-m	P_l	0	0	0	0	0	0	0	0	0
	P_d	719.1	719.1	0	689.1	689.1	0	173.6	173.6	0
	P_b	0	0	0	0	0	0	0	0	0

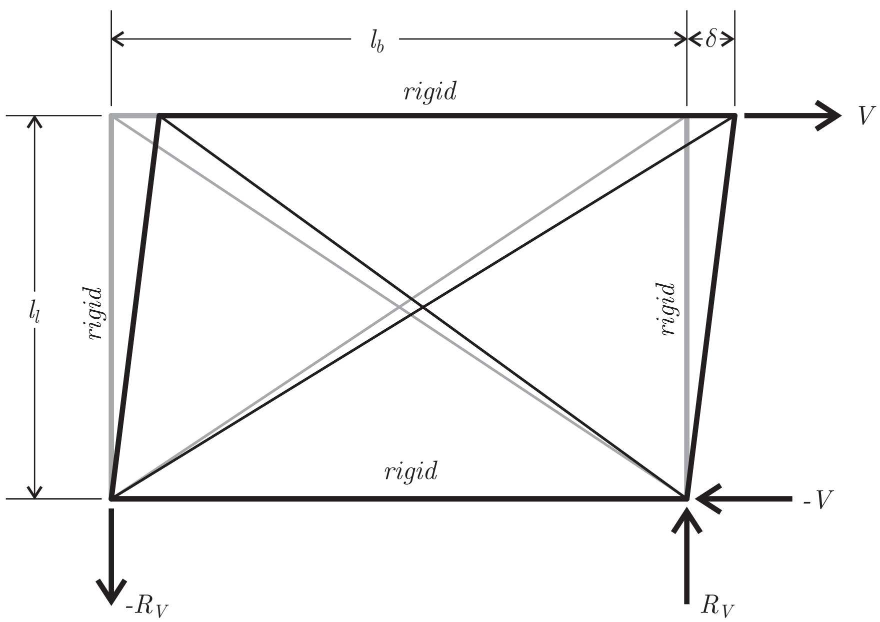

Figure 1: One truss face and associated diagonals, loads and deflections.

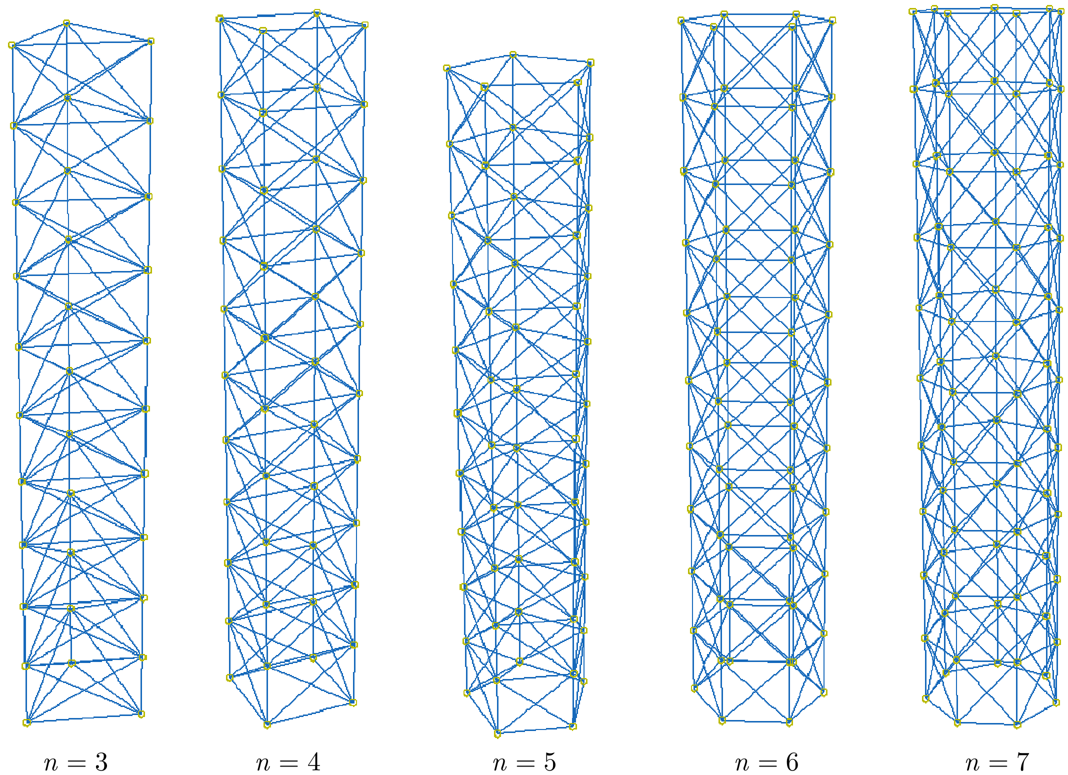

Figure 2: Truss finite element models for 3, 4, 5, 6 and 7 longeron trusses with two diagonals per truss face.